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A crossover in the scaling law of the Lyapunov exponent†

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Abstract. We consider products of random matrices appearing in the study of Schrödinger operators or dynamical systems. We show how a crossover in the scaling law of the Lyapunov exponent takes place when the standard deviation of the process increases.

1. Introduction

In this paper we proceed with the analysis of the 'growth of chaos' in a situation where a product of random symplectic matrices can be used as a good approximation for the tangent map of nonlinear Hamiltonian dynamics; see [1]. For applications to disordered systems, see [2] and [3].

We use the maximum Lyapunov exponent as the indicator of stochasticity as it measures the growth of exponential divergence of nearby orbits.

Such stochastic behaviour occurs in many physical situations. An important motivation for our paper was the intriguing problem of anomalous electron energy transport which is experimentally observed in magnetically confined plasmas. It has been suggested that low-frequency magnetic fluctuations could give an insight into this phenomenon. If this is the case, one important quantitative characteristic of the physical situation is the correlation length of the electron radial diffusion which is related [4] to the radial heat electron transport. It turns out that the inverse of the correlation length is precisely the Lyapunov exponent λ.

One of the surprising features of these measurements is that they agree with a scaling law $\lambda \sim \varepsilon^{2/3}$, where ε is the size of the nonlinear perturbation. Instead, mathematical models made with stochastic matrices predict a general scaling of power 1/2, the scaling of 2/3 being predicted only in the special case of the matrices of random elements with mean zero [1, 5, 6].

Here we give a possible explanation of this fact, based on the analysis of the scaling, both in the mean and in the variance of the process; we discover a universal scaling with respect to the variance after a critical value which is mean dependent. At this point a crossover takes place. In any case, after this critical value, the scaling in the size of the perturbation is the universal 2/3.

We remark [7] that for the magnetic instabilities in Tokomak plasmas, the variance of the process can also be related to such physical parameters as the location of the resonances. For this reason we think that our results are of interest *per se*.

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Finally we analyse this scaling in terms of the connectance of the matrices which, in turn, can be related to the interconnection of the different degrees of freedom of the problem.

Our analysis starts with an analytic treatment which is not very different from that of [6], except that the use of a tensor product formalism enables us to easily control the dependence in both the first and the second moment of the process.

Our numerical computations illustrate and confirm this analysis.

2. The power series

In this study we use the type of matrices appearing as tangent maps in dynamical systems. As it is known, we could also take the form of the matrices as they appear in the Schrödinger equation on a strip (see [8]), namely

$$\mathbb{R} = \begin{pmatrix} V & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

where the V are $n \times n$ matrices and $\mathbb{1}$ is the $n \times n$ identity, but the two formulations are equivalent.

So, we consider the symplectic maps

$$q_{m+1} = q_m + p_m$$
$$p_{m+1} = p_m + \varepsilon f(q_{m+1})$$

where $q_m, p_m \in \mathbb{R}^N$ and f is a smooth function and our matrices are the corresponding tangent maps; see below.

The first step of our computations is to derive an expression for the product of the matrices in a suitable form for further use.

We write

$$\mathbb{B}_M = \mathbb{A}_M \mathbb{A}_{M-1} \dots \mathbb{A}_1 \tag{1}$$

with the matrices $\mathbb{A}$ being symplectic, of the form

$$\mathbb{A}_i = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ A_i & 1 + A_i \end{pmatrix} \tag{2}$$

where $\mathbb{1}$ is the $N \times N$ identity and the A_i are $N \times N$ symmetric, independent and identically distributed, matrices.

In our case we define the maximal Lyapunov exponent (see [8]) as

$$\lambda = \lim_{M \to \infty} \frac{1}{M} \langle \log \|\mathbb{B}_M\| \rangle \tag{3}$$

where $\langle \cdot \rangle$ stands for the average on the process. If we specialise to the trace norm, (3) becomes:

$$\lambda = \lim_{M \to \infty} \frac{1}{2M} \langle \log \mathrm{Tr}(\mathbb{B}_M^* \mathbb{B}_M) \rangle. \tag{4}$$

The matrices A can be written:

$$\mathbb{A} = J \otimes \mathbb{1} + K \otimes A \tag{5}$$

where J and K are 2×2 matrices defined as

$$J=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \qquad K=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}. \tag{6}$$

Therefore $\mathbb{B}_M$ has the following form:

$$B_M=(J^M\otimes 1)+\sum_{p=1}^{\infty}\sum_{i_1+\ldots+i_{p+1}=M} J^{i_1}KJ^{i_2-1}K\ldots KJ^{i_{p+1}-1}\otimes A_{i_1+1}\ldots A_{i_p+i_{p-1}+\ldots+i_1+1} \tag{7}$$

where $i_1=0,1,2\ldots$ and $i_r=1,2,3\ldots$ for $r\neq 1$.

Now, since

$$KJ^rK=(r+1)K \tag{8}$$

and

$$K^r=K \tag{9}$$

we obtain:

$$\begin{aligned}\mathrm{Tr}(\mathbb{B}_M^*\mathbb{B}_M)=(M^2+2)+2\sum_{p=1}^{\infty}\sum_{i_1+\ldots+i_{p+1}=M}(i_2\ldots i_p)(i_1+Mi_1i_{p+1}+i_{p+1})\\ \times\mathrm{tr}(A_{i_1+1}\ldots A_{i_p+\ldots+i_1+1})+\sum_{p,q=1}^{\infty}\sum_{\substack{j_1+\ldots+j_{q+1}=M\\ i_1+\ldots+i_{p+1}=M}}(i_2\ldots i_p)(j_2\ldots j_q)\\ \times(i_1j_1+1)(i_{p+1}j_{p+1}+1)\mathrm{tr}(A_{j_q+\ldots+j_1+1}\ldots A_{j_1+1}A_{i_1+1}\ldots A_{i_p+\ldots+i_1+1}).\end{aligned} \tag{10}$$

Notice that (10) gives a complete separation, via the tensor product, of the deterministic part on the left and the random one on the right side.

In order to go further we need to interchange the mean procedure and the logarithm. Even if, in general, after this exchange we only get an upper bound for the Lyapunov exponent, thanks to the absence of intermittence, see [9], we expect a good agreement of this quantity with the Lyapunov exponent. Indeed this feeling is confirmed by numerical evidence, as will be clear later on, except for the region around the crossover for which there may be a difference between these two values.

3. The scaling law

First suppose that the matrices A_i are full, i.e.

$$A_i=(a_{k,r})_{k,r=1,\ldots,N} \tag{11}$$

all the $a_{k,r}$ are towings of the process, with the only constraint that $a_{k,r}=a_{r,k}$, and denote $\alpha=\langle a_{k,r}\rangle$ the common mean of the process.

Let us evaluate the first sum in (10).

We have

$$\langle\mathrm{tr}(A_{i_1+1}\ldots A_{i_p+\ldots+i_1+1})\rangle=\alpha^pN^p \tag{12}$$

and it is clear from the evaluation of the sum with constraint (see (4)) that we get the following estimate for the first sum:

$$\sum_{p=1}^{\infty} \alpha^p N^p \frac{M^{2p+1}}{(2p+1)!} \sim \exp(M\sqrt{\alpha N}) \tag{13}$$

where the non-exponential part was neglected in view of (10).

The estimation of the second term is more involved since the same matrix A may appear twice in the trace and therefore leads to correlations.

Let us consider

$$\mathrm{tr}(A_1 \dots A_n) \tag{14}$$

where the matrices $A_1 \dots A_n$ are one of the possible choices in the last term of (10), i.e. some of them appearing twice, but no more. Let r be the number of repeated matrices and $(n-2r)$ the remaining independent matrices in this product. Let $\{\xi_k\}$, $k = 1, \dots, N$ be the canonical bases of $\mathbb{R}^N$ and denote by

$$(k_1, k_2, \dots, k_n) \qquad 0 < k_i \leqslant N \tag{15}$$

the product

$$(\xi_{k_1}, A_1\xi_{k_2})(\xi_{k_2}, A_2\xi_{k_3}) \dots (\xi_{k_n}, A_n\xi_{k_1}). \tag{16}$$

The trace (15) should be the sum of all the possible chains (16). Each of the chains gives a contribution of

$$\gamma^s \alpha^{n-2s} \qquad s = 0, 1, \dots, r \tag{17}$$

where $\gamma = \langle a_{u,v} a_{u,v} \rangle$.

In order to evaluate the number of chains of each type we take the two extreme cases: first, all the $2r$ repeated matrices are separated, one from the other, by at least one of the $(n-2r)$ remaining matrices; the second case considered is that where all the $2r$ matrices are adjacent.

Let us consider the former case: the number of chains giving a contribution $\gamma^s\alpha^{n-2s}$, $0 \leqslant s \leqslant r$, is

$$\binom{r}{s} N^{n-2r} 2^s (N^2-2)^{r-s} \tag{18}$$

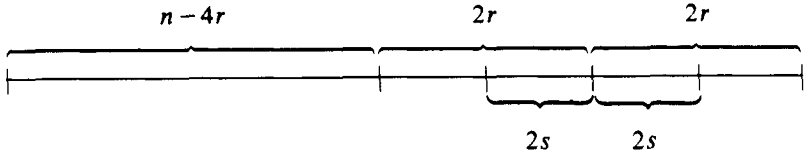
where the order of the index is irrelevant, and we see that the first $n-4r+2r = n-2r$ (on the left) values of the index in the chain can be freely choosen among the N possibilities. Each of the 2^s possibilities of the following s pairs gives a contribution of order γ. Finally for the last $(r-s)$ pairs the choice is free except for one of the two possibilities that would increase the exponent of γ. Therefore there are (N^2-2) possibilities for each pair. The meaning of the combinatorial term is clear.

We end up in this case with the following expression:

$$\mathrm{tr}(A_1 \dots A_n) = N^n \alpha^n \left(\frac{N^2-2}{N^2} + \frac{2\gamma}{N^2\alpha^2} \right)^r. \tag{19}$$

In the opposite case, where all the $2r$ matrices are adjacent we can only perform an upper bound since we avoid the combinatory of all the possible positions of the s repeated matrix elements inside the part of the chain corresponding to the $(r+1)$ indexes. The upper bound is obtained where all the s matrices are adjacent.

Indeed we can represent such a chain in the following way:



The number of different chains giving a contribution of $\gamma^s\alpha^{n-2s}$, $s=0,1,\ldots,r$, is therefore bounded by

$$\binom{r}{s} N^{n-r}N^{r-s} \tag{20}$$

and therefore we obtain the following bound:

$$\operatorname{tr}(A_1\ldots A_n) \leqslant N^n\alpha^n\left(1+\frac{\gamma}{N\alpha^2}\right)^r. \tag{21}$$

If we suppose that the matrices A have a connectance N_c, i.e. $a_{k,r}=0$ if $|k-r|>N_c$, except for $a_{1,N}=a_{N,1}$, then a good approximation for either (19) or (21) is to replace N by the mean of the number of occupied entrances in the matrix, $\bar{N}$, namely

$$\bar{N} = 1+2N_c-\frac{N_c(N_c+1)}{N}+2\frac{\delta}{N} \tag{22}$$

where $\delta=0$ if $(N-N_c)\leqslant 1$ or $N=1$ and otherwise $\delta=1$.

Now we come back to (10) in order to estimate the last summation.

We first perform a change of variables in order to be able to use (19) and (21). Writing

$$\begin{aligned} &i_1+i_2+\ldots+i_k=\hat{i}_k && k=1,2,\ldots,p+1\\ &j_1+j_2+\ldots+j_r=\hat{j}_r && r=1,2,\ldots,q+1 \end{aligned} \tag{23}$$

the constraint on the sum in (10) is now

$$\begin{aligned} &0\leqslant \hat{i}_1\leqslant \hat{i}_2\leqslant\ldots\leqslant \hat{i}_p\leqslant M\\ &0\leqslant \hat{j}_1\leqslant \hat{j}_2\leqslant\ldots\leqslant \hat{j}_q\leqslant M. \end{aligned} \tag{24}$$

In view of the above computation, a typical term of the sum is such that r of indexes $\hat{i}$ coincide with r of the $\hat{j}$ leading to a value of the trace estimated in (19) and (21) where, clearly, $n=p+q$ (for simplicity of notation we begin the summation at $p=q=0$ in order to avoid $p+1$ and $q+1$ in the formula).

Now, in this case, the estimation of the trace on the right side of the tensor product is independent of the values of the indexes $\hat{i}$ and $\hat{j}$, provided the coincidence condition for r of the indexes is fulfilled and we need only to estimate a sum of the type:

$$\sum_{\substack{\hat{i}_1\leqslant\ldots\leqslant\hat{i}_p\leqslant M\\ \hat{j}_1\leqslant\ldots\leqslant\hat{j}_q\leqslant M\\ \hat{i}_1=\hat{j}_1,\ldots,\hat{i}_r=\hat{j}_r}} (\hat{i}_1,\ldots,\hat{i}_p)(\hat{j}_1,\ldots,\hat{j}_q). \tag{25}$$

The sum (25) is only a typical term, since on the one hand there are also terms with $(p-1)$ instead of p and $(q-1)$ instead of q, and on the other hand, due to the change of variables (23), there are also terms where some of the $\hat{i}$ or $\hat{j}$ are squared but where at the same time the total number p or q is decreased in the same way. Nevertheless it will be clear from the following that (25) gives the leading term in powers of M. The fact that we take the first (and no other) r indexes to be coincident does not essentially change in the estimation as we shall see.

We evaluate (25) via the integral (since M is large):

$$\int_0^M \int_0^{\hat{i}_1} \ldots \int_0^{\hat{j}_{q-1}} \hat{i}_1^2 \ldots \hat{i}_r^2 \hat{i}_{r+1} \ldots \hat{i}_p \hat{j}_{r+1} \ldots \hat{j}_q \, \mathrm{d}\hat{i}_1 \ldots \mathrm{d}\hat{i}_r \, \mathrm{d}\hat{i}_{r+1} \ldots \mathrm{d}\hat{j}_q$$

$$\approx 2^r M^{2(p+q)-r+3} \left(\frac{1}{(2p+r+2)!} \frac{1}{2(q-r)!} + \frac{1}{(2q+r+2)!} \frac{1}{2(p-r)!} \right)$$

$$= C(p,q,r). \tag{26}$$

We therefore write the last term of (10) as

$$2 \sum_{p=0}^{\infty} \sum_{q=0}^{p} \sum_{r=0}^{q} C(p,q,r) N^{p+q} \alpha^{p+q} \chi^r \tag{27}$$

where χ is in between the correspondent expressions in (19) and (21).

Notice, that, for each value of q we get a polynomial of degree r in χ.

Now there are two extreme cases: if γ is small enough the dominant term of the polynomial is for $r = 0$ and then the leading term in powers of M is obtained for $p = q$ and we recover a term of the same type as (13) and the known scaling law in $(\alpha)^{1/2}$. Notice that we also discover a scaling law in $(\bar{N})^{1/2}$, and indeed our numerical computation (see figure 1) confirms this property. In the opposite direction, if γ is large enough, the dominant term in the polynomial is for $r = q$, and again in this case the leading term in powers of M is obtained for $p = q$.

Under this condition we obtain

$$C(p,p,p) \approx \frac{M^{3p+2}}{(3p+2)!} 2^p \tag{28}$$

and

$$\alpha^{2p} \chi^p = \left(\frac{\bar{N}^2 - 2}{\bar{N}^2} \alpha^2 + \frac{2\gamma}{\bar{N}^2} \right)^p$$

or

$$\alpha^{2p} \chi^p = \left(\alpha^2 + \frac{\gamma}{\bar{N}} \right)^p \tag{29}$$

according to whether we take (19) or (21).

Denoting $\bar{\gamma}^p$ for the right term, with $\bar{\gamma}$ linear in γ, between the two expressions in (29), we finally obtain, combining (27), (28) and (29), the following estimation for the last part of (10):

$$\sim \exp(\tfrac{1}{2} \bar{N}^{2/3} 2^{1/3} \bar{\gamma}^{1/3} M). \tag{30}$$

If $\bar{\gamma} \gg \alpha$, the first sum in (10) gives no contribution to the Lyapunov exponent, according to (9) and we end up with a scaling law of type

$$\lambda(\gamma) \sim \bar{N}^{2/3} \bar{\gamma}^{1/3}. \tag{31}$$

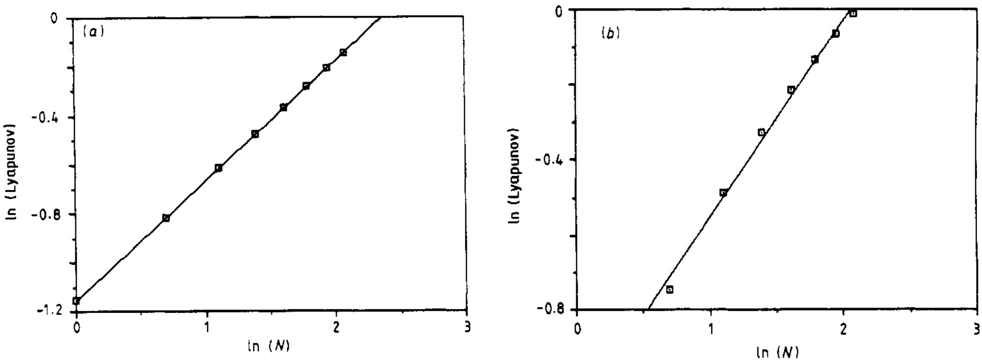


Figure 1. Lyapunov exponent as function of N (log-log scale). (a) $\alpha=0.1$ and $\gamma=0.01$; (b) $\alpha=0.001$ and $\gamma=0.64$.

Remark. Since $\bar{\gamma}\propto\varepsilon^2$ for a process of size ε, i.e. with matrices of the form

$$\mathbb{A}=\begin{pmatrix}\mathbb{1} & \mathbb{1}\\ \varepsilon A & \mathbb{1}+\varepsilon A\end{pmatrix} \tag{32}$$

we get in this case a scaling law with exponent 2/3 as it is the case for $\alpha=0$, provided the variance of the non-perturbed process is large enough.

4. Numerical experiments

We have performed a large number of numerical computations in order to test the approximations and the results of the previous sections. We use the uniform or Gaussian distributions of the NAG library after the rescaling of α and γ. The computation is made with double precision and with a stabilisation test. The number of iterations needed in any case is between 6×10^4 and 1×10^7. First of all we compute the Lyapunov exponents for small standard deviation γ as a function of the mean value α of the process. Our computation shows that the Lyapunov exponent in this case (small values of γ) is exactly

$$\lambda=\sqrt{\bar{N}\alpha} \tag{33}$$

where $\bar{N}$ is the mean number of degrees of freedom, given by formula (22).

Notice that this result agrees with (13). Moreover the substitution of N by $\bar{N}$ if the matrices are not full and the interchange of the mean procedure and the logarithm (see section 2) are in very good agreement with the numerical results. The latter seems to indicate the absence of intermittence in this case, since the Lyapunov and generalised Lyapunov exponents coincide. We have made tests for $N=1,\ldots,6$ and the corresponding possibilities of N_c. Figure 1(a) shows the Lyapunov exponent as a function of N (in a log-log scale) in the region $\gamma<\gamma_c$ and figure 1(b) the corresponding plot for $\gamma>\gamma_c$. As is clear, the Lyapunov exponent then has the following thermodynamic limit ($N\to\infty$ and N_c constant):

$$\lambda=\sqrt{2N_c+1}\,\sqrt{\alpha}. \tag{34}$$

We have also tested the case $\alpha=0$ (see [6]) but here we notice a difference between (31) and the computation of the Lyapunov exponent. A possible explanation for that is the interchange of the mean procedure and the logarithm, made in the derivation of (31); see section 2. Indeed, the difference between these two quantities increases with N, showing how fluctuations grow.

Finally we have tested the crossover. In order to do this we fixed α and we measured the Lyapunov exponent when γ changes, for several values of N and $\tilde{N}$ (N from 1 to 5 with all possible N_c). Figure 2 shows two plots of the Lyapunov exponent as a function of the standard deviation γ. For each fixed α, notice the 'plateau' corresponding to the range of values of γ for which the dominant term is as in (13) and therefore being constant. After a critical value γ_c, depending on α, the new scaling law clearly

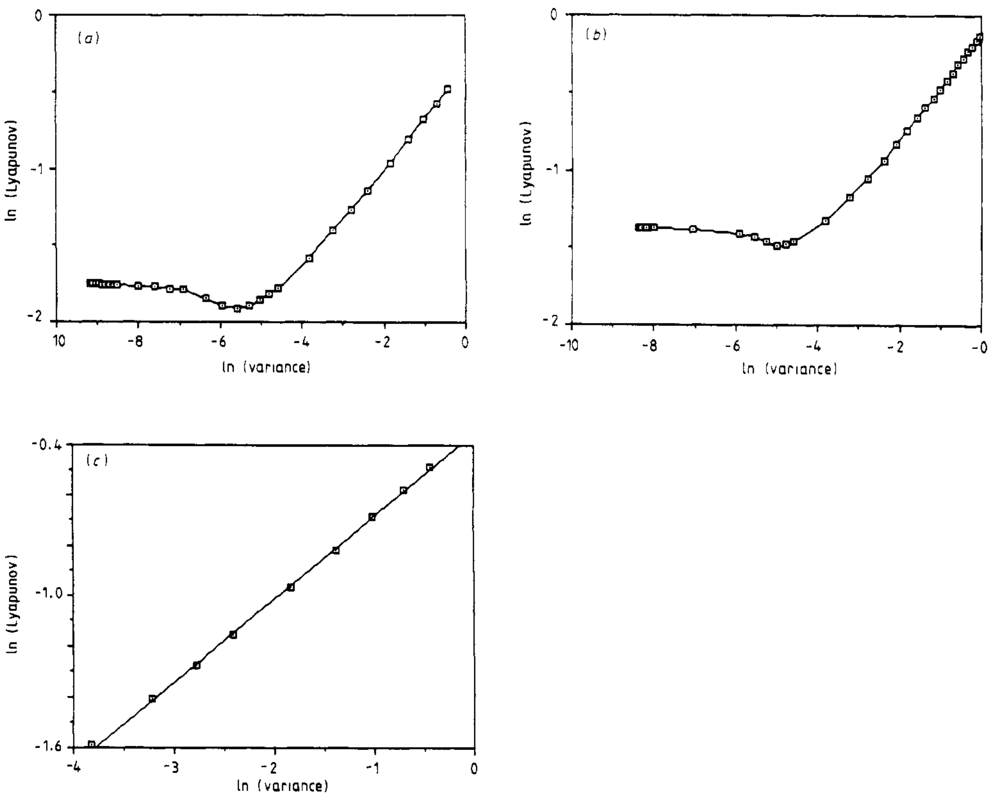


Figure 2. Lyapunov exponent as function of γ for fixed values of α (log-log scale): the crossover. (a) $\alpha = 0.01$ and $N = 3$; (b) $\alpha = 0.01$ and $N = 5$; (c) Particular of (a) for $\gamma > \gamma_c$ showing the 1/3 slope.

emerges and we can see the law

$$\lambda \sim \gamma^{1/3}. \tag{35}$$

We have also made some tests for the dependence on N and large values of γ. It seems that a scaling law as $N^{1/2}$ appears, which is compatible with (19) and (21) but we could not avoid fluctuations when γ is changed.

Finally, in figure 3 we show γ_c as a function of the mean α, for two different values of N.

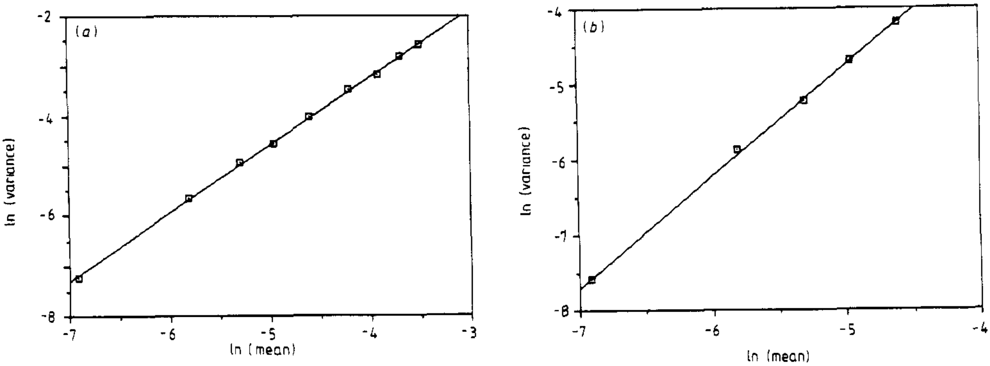


Figure 3. Critical values of γ as function of α. (*a*) $N=3$, $N_c=1$; (*b*) $N=5$, $N_c=2$.

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